On a Generalization of Alexander Polynomial for Long Virtual Knots

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Abstract

We construct new invariant polynomial for long virtual knots. It is a generalization of Alexander polynomial. We designate it by ζ meaning an analogy with ζ-polynomial for virtual links. A degree of ζ-polynomial estimates a virtual crossing number. We describe some application of ζ-polynomial for the study of minimal long virtual diagrams with respect number of virtual crossings.

Virtual knot theory was invented by Kauffman around 1996 [Ka1]. Long virtual knot theory was invented in [GPV] by M. Goussarov, M. Polyak, and O. Viro. ζ-polynomial for virtual link was introduced independently by several authors (see [KR],[Saw],[SW],[Ma1]), for the proof of their coincidence, see [BF]. The idea of two types of classical crossings in a long diagram, which were called $\circ$ (circle) and $*$ (star), was invented by V.O. Manturov (see [Ma4],[Ma3]). In present paper we called $\circ$ and $*$ crossings by *early overcrossing* and *early undercrossing* respectively. To consider early overcrossings and early undercrossings is the basis idea for a construction of ζ-polynomial in the case of long virtual knots.

Definition 1.1. By a *long virtual knot diagram* we mean a smooth immersion $f : \mathbb{R} \to \mathbb{R}^2$ such that:
1) outside some big circle, we have $f(t) = (t, 0)$;
2) each intersection point is double and transverse;
3) each intersection point is endowed with classical (with a choice for underpass and overpass specified) or virtual crossing structure.

Definition 1.2. A *long virtual knot* is an equivalence class of long virtual knot diagrams modulo generalized Reidemeister moves.

Definition 1.3. By an *arc* of a long virtual knot diagram we mean a connected component of the set, obtained from the diagram by deleting all virtual crossings (at classical crossing the undercrossing pair of edges of the diagram is thought to be disjoint as it is usually illustrated).

Definition 1.4. We say that two arcs a, a' belong to the same *long arc* if there exists a sequence of arcs $a = a_1, \dots, a_{n+1} = a'$ and virtual crossings $c_1, \dots, c_n$ such that for $i = 1, \dots, n$ the arcs a_i, a_{i+1} are incident to c_i from opposite sides.

Throughout the paper, we mean that initial and final long arcs, γ_- and γ_+, form united long arc $\gamma = \gamma_- \cup \gamma_+$. Let D be a long virtual diagram with $n \geqslant 1$ classical crossings. Hence, there is a natural pairing of all classical crossings and all long arcs: classical crossing v and long arc γ, which emanates from v, are paired.

We say that classical crossing v is *early overcrossing* (*early undercrossing*) if we have an arc passing over (under) v at first, in the natural order on long virtual diagram (see also [KM], p. 139).

Definition 1.5. An *incidence coefficient* $[v : a] \in T = \mathbb{Z}[p, p^{-1}, q, q^{-1}]/((p-1)(p-q), (q-1)(p-q))$ of classical crossing v and arc a is defined as a sum of some of three polynomials: $[v : a] = \varepsilon_1 1 + \varepsilon_2(t^{sgn\, v} - 1) + \varepsilon_3(-t^{sgn\, v})$, where $\varepsilon_i \in \{0, 1\}, i = 1, 2, 3$; $t = p$ if v is early overcrossing, $t = q$ if v is early undercrossing; $sgn\, v$ denotes *local writhe number* of v. We set $\varepsilon_1 = 1 \Leftrightarrow$ arc a is emanating from v; $\varepsilon_2 = 1 \Leftrightarrow a$ is passing over v; $\varepsilon_3 = 1 \Leftrightarrow a$ is coming into v. If v and a are not incident we set $[v : a] = 0$.

Let us enumerate all classical crossings of D by numbers $1, ..., n$ in arbitrary way and associate with each classical crossings the emanating long arc. Our generalization of Alexander polynomial for long virtual knots is defined as determinant of $n \times n$-matrix $A(D)$ with elements

$$A_{ij} := \sum_{a \subset \gamma^j} [v_i : a] s^{deg\, a} \in T[s, s^{-1}]$$

The function deg : {arcs of D} $\to \mathbb{Z}$ is defined according to the rules:
(1) if arc a is a first at a long arc, $deg\, a = 0$;
(2) if arcs a and b are neighbour on a long arc, a precedes b, then $deg\, b = deg\, a + 1$, if we pass from the left to the right with respect to the transversal arc, and $deg\, b = deg\, a - 1$ otherwise. In the first case we called such virtual crossing *increasing*, in the second case — *decreasing*.

It easy to see that polynomial $\zeta(D) = det\, A(D)$ does not depend on a numeration of classical crossings.

By analogy with [AM] we formulate following three theorems.

Theorem 1.1. *If virtual diagrams D, D' are equivalent then $\zeta(D') = q^r \zeta(D)$ for some integer r.*

A sketch of the proof. The invariance of ζ for Reidemeister moves $\Omega_1', \Omega_2', \Omega_3'$ is evident. The checking of invariance for Ω' and Ω_2 is similar to the case of ζ-polynomial for virtual link (see [Ma2],[Ma3]).

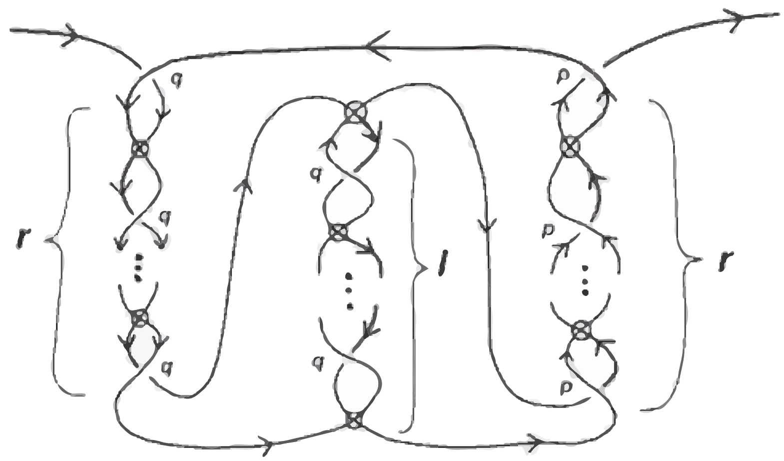

Figure 1: Long knot $D_{r,l}$, $r, l \geqslant 0$

There are two types of the first Reidemeister move Ω_1: Ω_1^p, if we have early overcrossing, and Ω_1^q, if we have early undercrossing. It easy to calculate that $\zeta(\Omega_1^p(D)) = \zeta(D)$, $\zeta(\Omega_1^q(D)) = q^{\pm 1}\zeta(D)$.

It is convenient to use the Laplace theorem (about determinants) to check that $det\, A(\Omega_3(D)) = det\, A(D)$. We check equality for 10 pair of 3×3-minors of matrices $A(\Omega_3(D))$ and $det\, A(D)$. Two of these pairs give equalities only if we set $(p-1)(p-q) = 0, (q-1)(p-q) = 0$.

□

Theorem 1.2. *Let k be the number of virtual crossings on a long virtual diagram D. Then $deg_s\, \zeta(D) \leqslant k$.*

From Theorems 1.1 and 1.2 we easily conclude

Corollary 1.1. *If $deg_s\, \zeta(D) = k$ then D has minimal virtual crossing number.*

For checking of minimality by using Corollary 1.1 it is convenient to use

Theorem 1.3. *The s^k-th coefficient of $\zeta(D)$ is equal to $det\, B$, where $B_{ij} = [v_i : a_j]$ if $\exists\, a_j \subset \gamma^j$ s.t. $deg\, a_j =$#of increasing virtual crossings on γ^j, and $B_{ij} = 0$ otherwise, $i, j = 1, ..., n$.*

Example. In **Figure** we draw long virtual diagram $D_{r,l}$ which closure is unknot. Arcs a_j, $j = 1, ..., n$, are marked by thick lines. By Theorem 1.3 the s^k-th coefficient of $\zeta(D)$ is equal to $|[v_i : a_j]|_{i,j=1,...,n} = q^{r+l}(qp^{-1}-1) = q-p \neq 0$ in the ring T. Consequently, $D_{r,l}$ is minimal by Corollary 1.1.

By using our ζ-polynomial we can proof following Conjecture in a particular case. Here symbol $*$ denotes usual product of long knots.

Conjecture. *If D is a minimal long virtual diagram with respect number of virtual crossings, K is a long classical knot diagram, then $D * K$ is also minimal.*

Theorem 1.4. *(the particular case of Conjecture)*
If D is a minimal long virtual diagram s.t. $deg_s\,\zeta(D)$ is equal to virtual crossing number of D, K is a long classical knot diagram, then $D * K$ is minimal.

For a proof of Theorem 1.4 we use following lemmas. Let l be a number of long arc $\gamma = \gamma_- \cup \gamma_+$, where γ_- and γ_+ are initial and final long arcs respectively. Then $A_{il} := \sum_{a\subset\gamma}[v_i : a]s^{deg\,a} = \sum_{a\subset\gamma_-}[v_i : a]s^{deg\,a} + \sum_{a\subset\gamma_+}[v_i : a]s^{deg\,a}$. Consequently, $det\,A(D) = det\,A^-(D) + det\,A^+(D)$, where $A^{\pm}_{il} = \sum_{a\subset\gamma_\pm}[v_i : a]s^{deg\,a}$, $A^{\pm}_{ij} = A_{ij}$ for $j \neq l$. Thus, we have the natural decomposition of ζ-polynomial: $\zeta(D) = \zeta_-(D) + \zeta_+(D)$, where $\zeta_\pm(D) := det\,A^\pm(D)$.

Lemma 1.1. $\zeta_-(D_1 * D_2) = -\zeta_-(D_1)\zeta_-(D_2)$*;* $\zeta_+(D_1 * D_2) = \zeta_+(D_1)\zeta_+(D_2)$.

Lemma 1.2. $x \in T = \mathbb{Z}[p, p^{-1}, q, q^{-1}]/((p-1)(p-q), (q-1)(p-q))$ *is zero divisor* $\Leftrightarrow x|_{p=1,\,q=1} = 0$.

Proof of Theorem 1.4. By Lemma 1.1 $\zeta(D * K) = \zeta_-(D * K) + \zeta_+(D * K) = -\zeta_-(D)\zeta_-(K) + \zeta_+(D)\zeta_+(K) = \zeta_+(K)\zeta(D)$, because $\zeta(K) = 0$. Consequently, $deg_s\,\zeta(D * K) = deg_s\,\zeta(D)$ if $\zeta_+(K) \in T$ is not zero divisor.

It easy to check that $\zeta_+(K)|_{p=1,\,q=1} = \pm\Delta(K)|_{t=1}$, where Δ denotes Alexander polynomial. It is known that $\Delta(K)|_{t=1} = \pm 1$. Hence, by Lemma 1.2 $\zeta_+(K)$ is not zero divisor, because $\zeta_+(K)|_{p=1,\,q=1} \neq 0$. □

Acknowledgements

The author is grateful to V.O. Manturov for idea of ζ-polynomial for long virtual knots and fruitful consultations.

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